An Axiomatic Approach to Ranking Systems

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Ranking Systems – Introduction

- Systems in which agents ranks for each other are aggregated into a social ranking.
- Examples:

PageRank

Reputation System

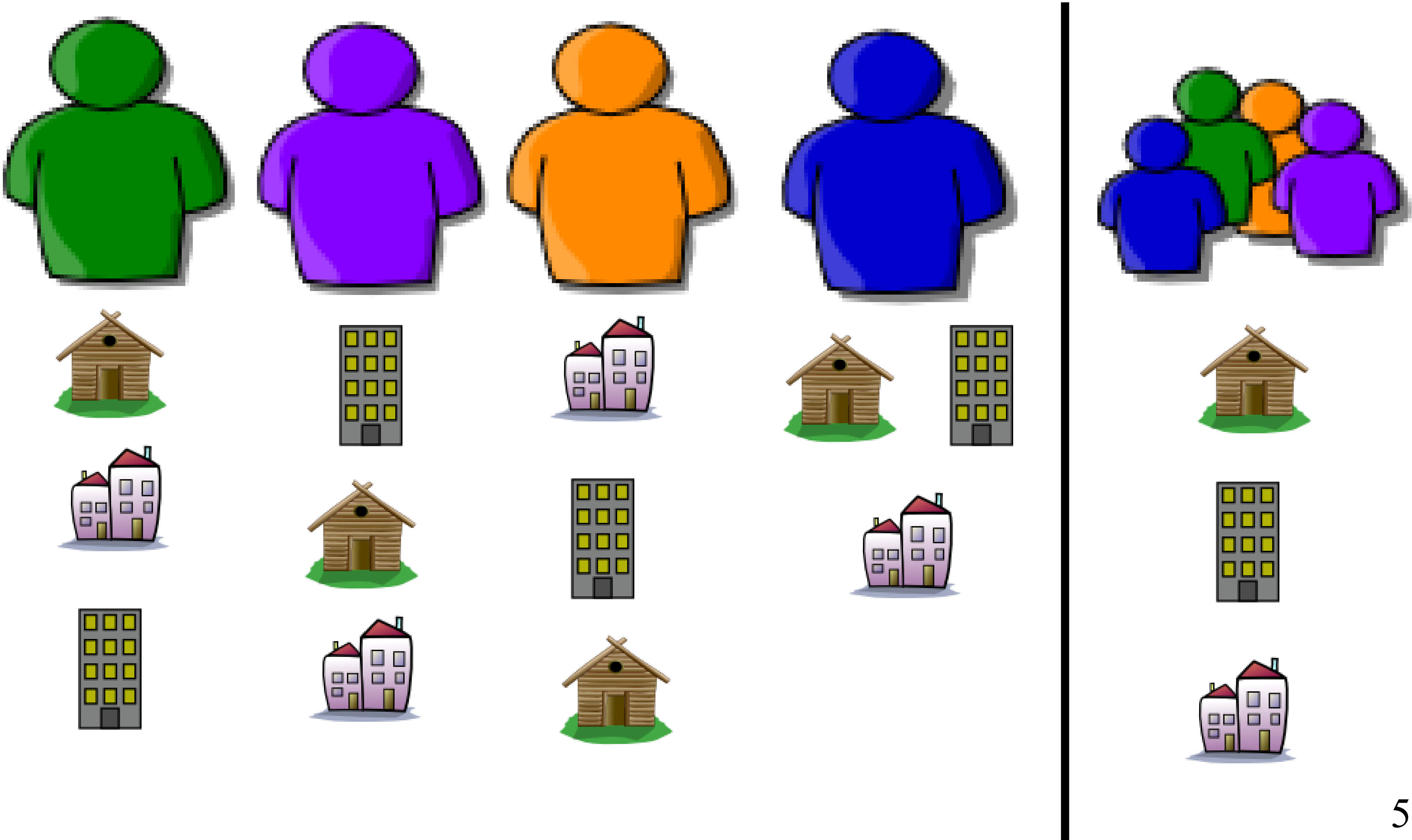
Ranking Systems

- Ranking systems can be defined in the terms of a *ranking function* combining the individual votes of the agents into a social ranking of the agents.
- Can be seen as a variation of the *social choice* problem where the agents and alternatives coincide.

Social Choice

- The classical *social choice* setting is comprised of:
  - A set of agents
  - A set of alternatives
  - A preference relation for each agent over the set of alternatives.
- A *social welfare function* is a mapping between the agents' individual preferences into a social ranking over the alternatives.
- The goal: produce “good” social welfare functions.

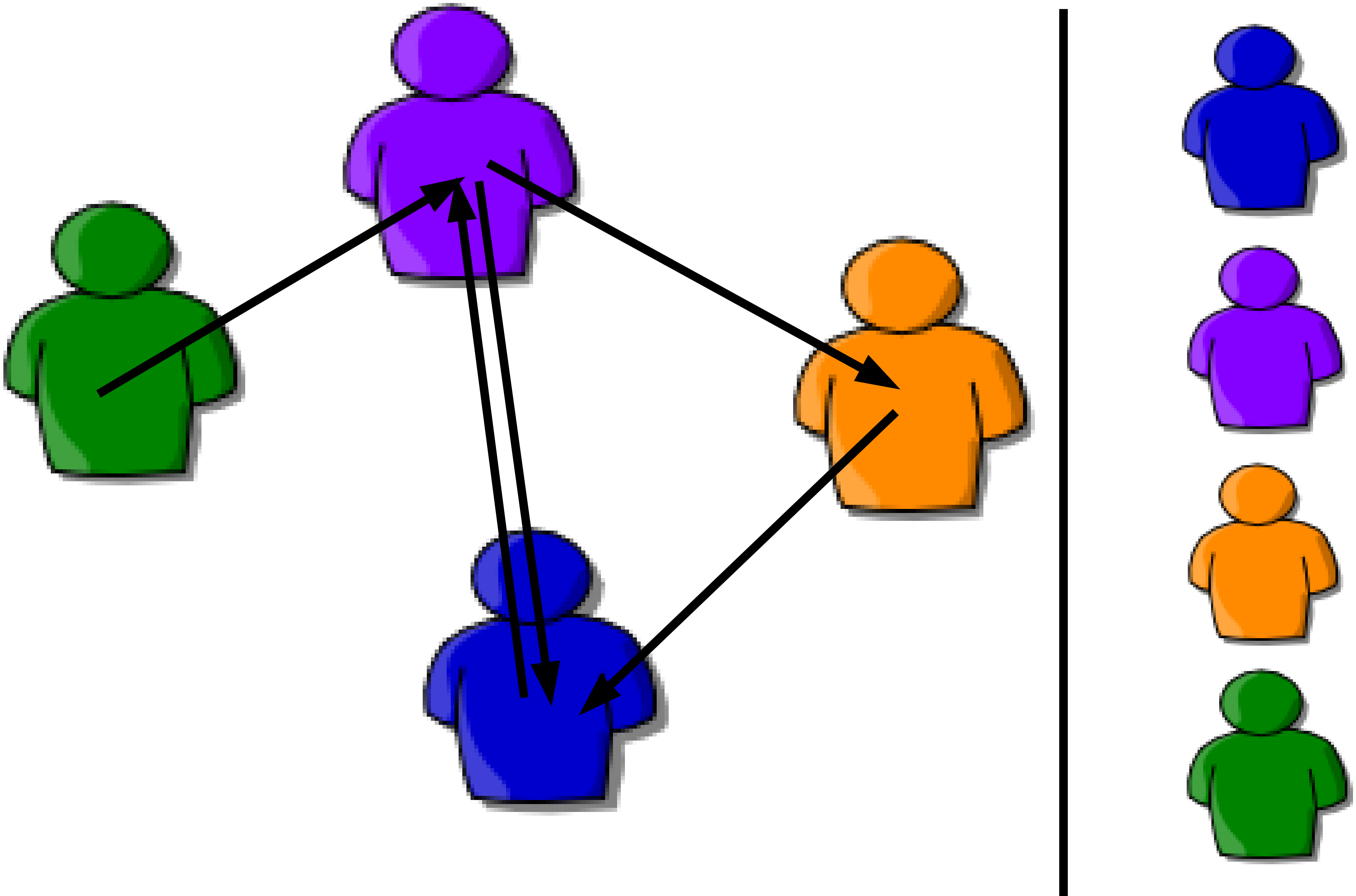
Social Choice - Example



Graph Ranking Systems

- Voters and alternatives are the same set.
- Each agent may only make binary votes: only specify some subset of the agents as “good”.
- Preferences of all the agents may be represented as a graph, where the agents are the vertices and the votes are the edges.
- Applies for ranking WWW pages and eBay traders.

Ranking systems



Ranking System – Definition

- Therefore, a (graph) *ranking system* can simply be defined as a functional from the set of all graphs, to the set of linear orderings on the vertices.
- Such a function may be partial. That is, rank only a specific set of graphs, in which case we call it a *partial ranking system*.

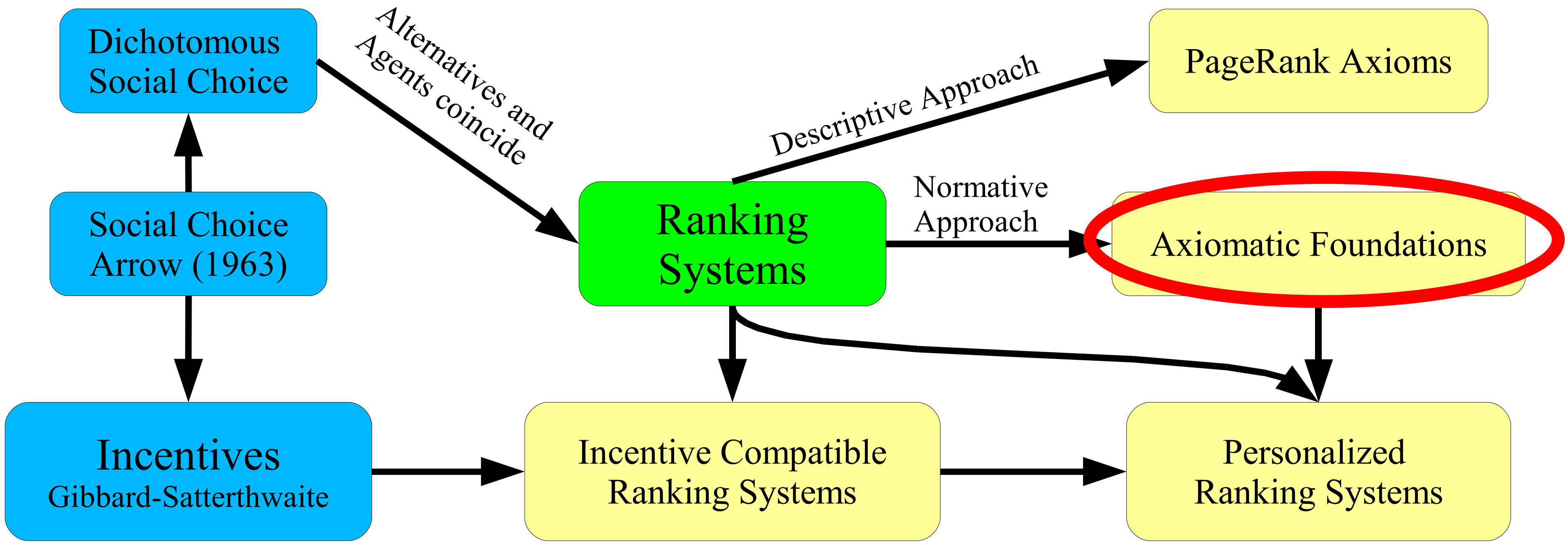
The Axiomatic Approach

- We try to find basic properties (**axioms**) satisfied by ranking systems.
- Encompasses two distinct approaches:
  - The normative approach, in which we study sets of axioms that *should* be satisfied by a ranking system; and
  - The descriptive approach, in which we devise a set of axioms that *are* uniquely satisfied by a known ranking system
- We apply both to ranking systems

Why the Axiomatic Approach?

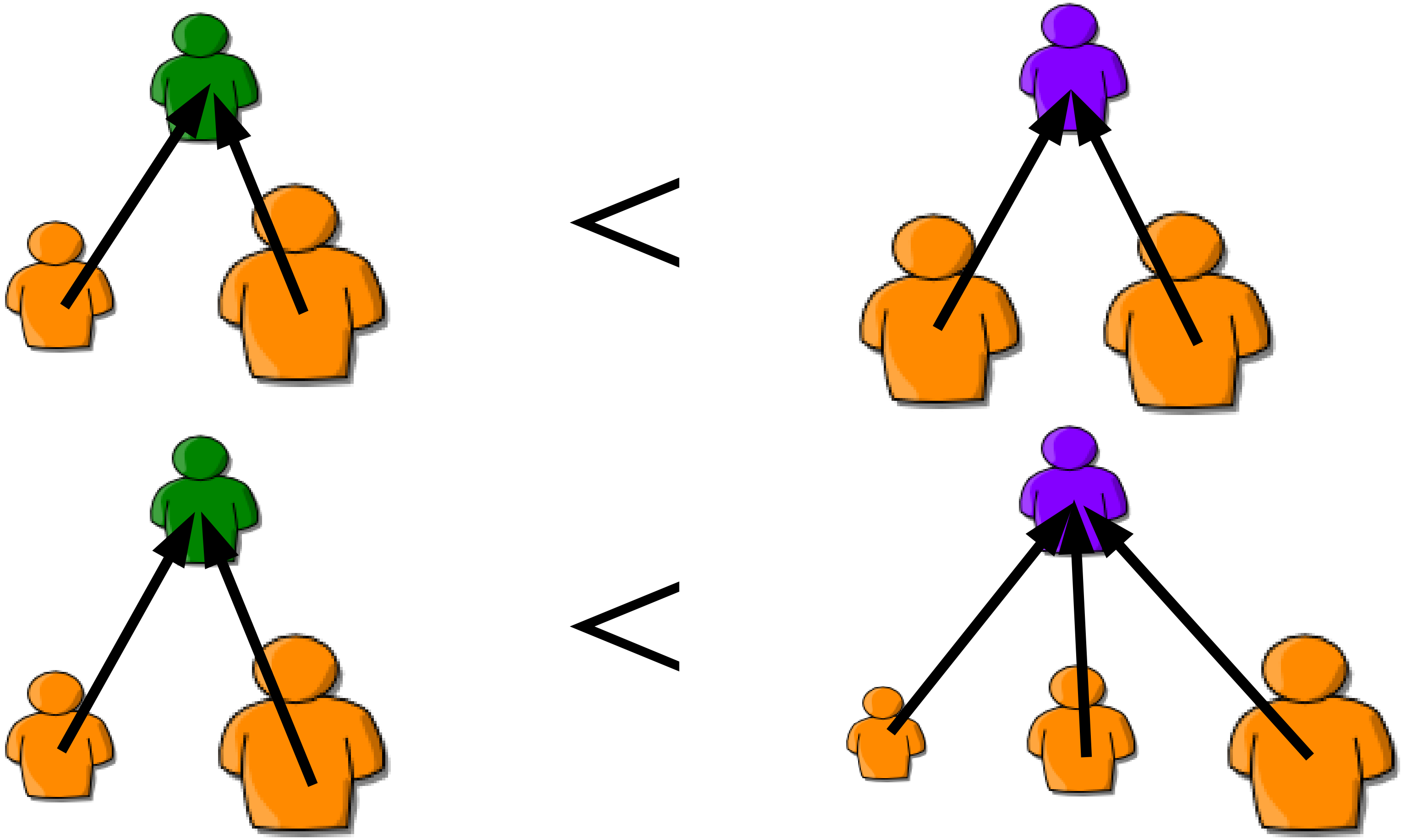
- Better understanding
  - Axiomatic analysis lets us understand ranking systems in terms of features they posses.
- Objective evaluation
  - Axiomatic analysis of ranking systems gives us an objective measure of quality for ranking systems.
- Understanding limitations
  - Impossibility results allow us to limit our search for new ranking systems.

Research Map



Transitive Effects

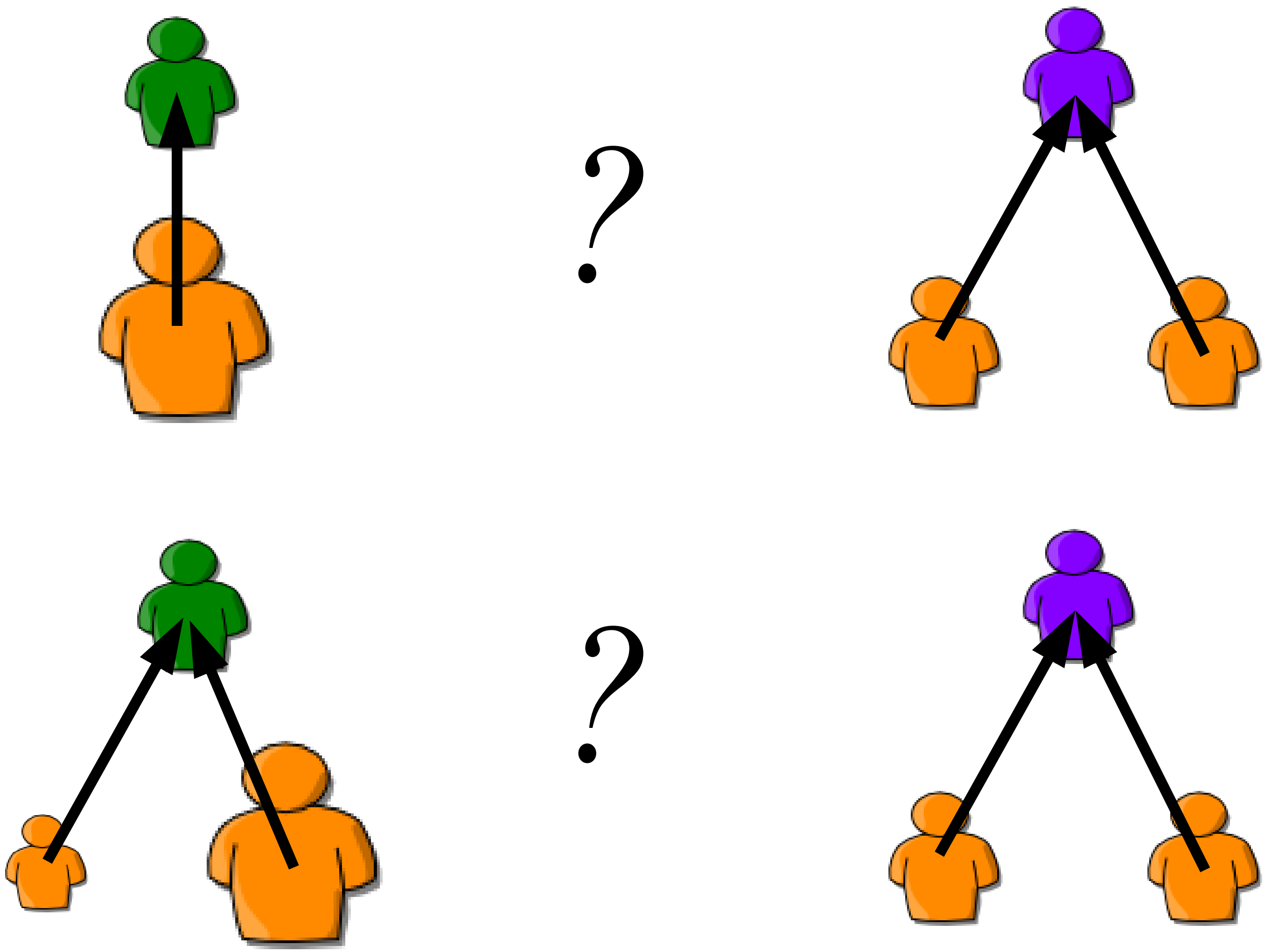
The rank of your voters should affect your own.



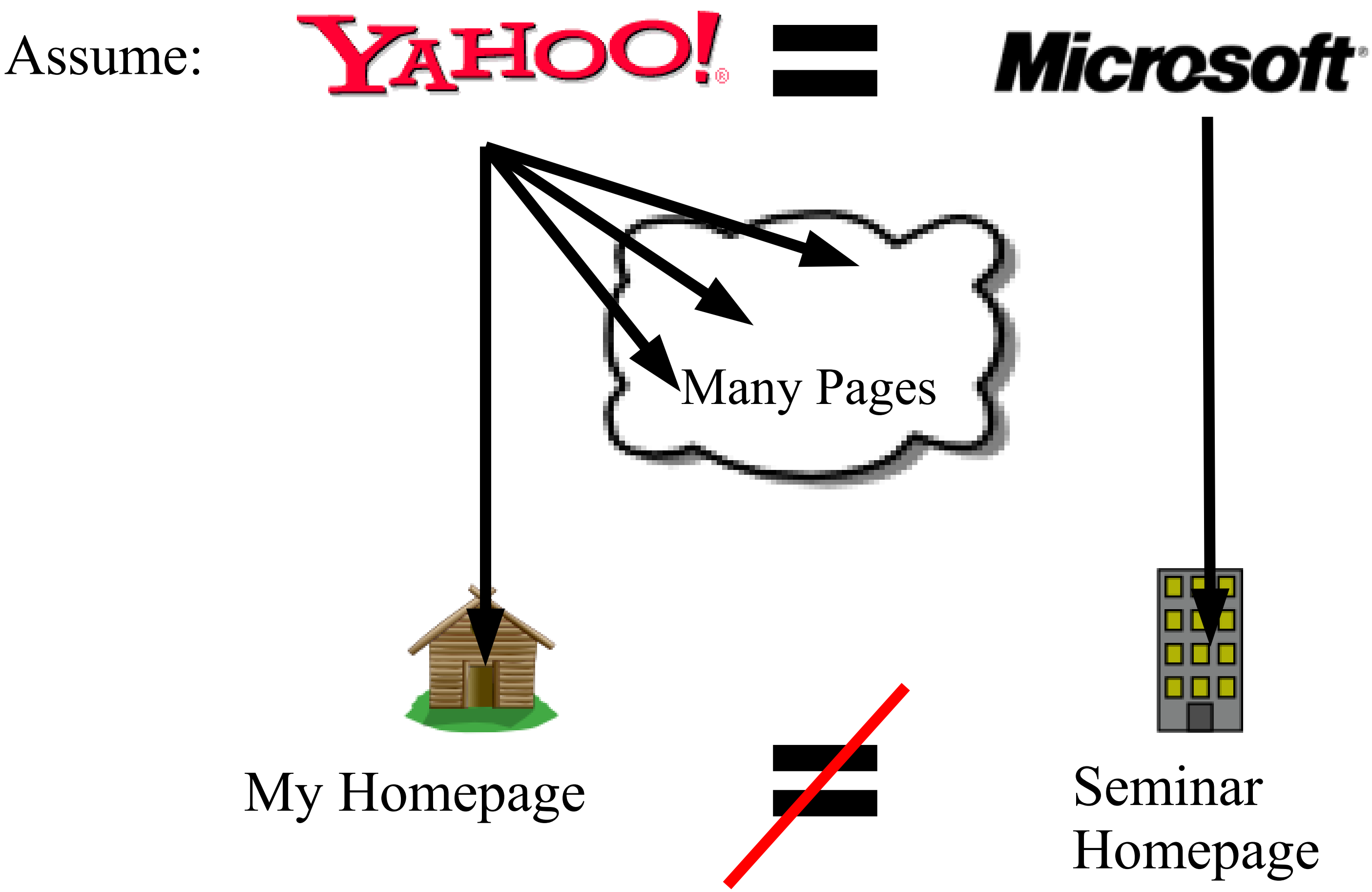
Strong Transitivity

- Formally, a ranking system *F* satisfies *strong transitivity* if for every two vertices *x,y* where *F* ranks *x*'s predecessor set *P(x)* is (strictly) weaker than *P(y)*, then *F* must rank *x* (strictly) weaker than *y*.
- We define a predecessor set *P(x)* as being weaker than *P(y)* as the existence of a 1-1 mapping between *P(x)* and *P(y)* where every vertex in *P(x)* is mapped to a stronger or equal vertex in *P(y)* and at least one of the comparisons is strict, or the mapping is not onto.

Strong Transitivity Doesn't always apply



Strong Transitivity too Strong?

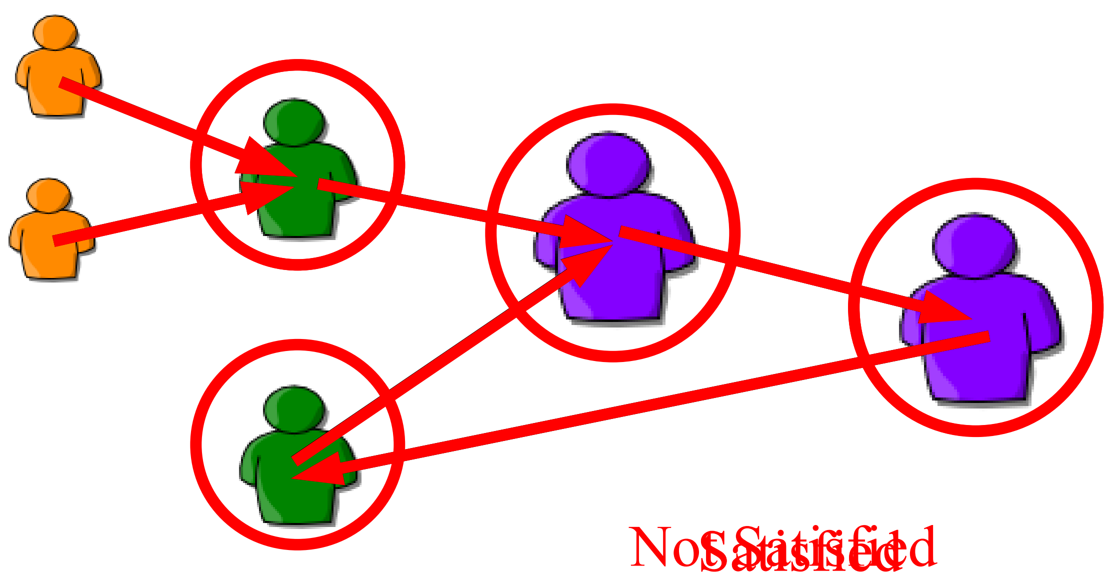


More about Transitivity

- Weak Transitivity
  - The idea: Only match predecessors with equal out-degree.
  - We assume nothing about predecessors of different out-degrees.
  - Otherwise, same as Strong Transitivity.
- PageRank satisfies Weak Transitivity but not Strong Transitivity.
- Strong Transitivity can be satisfied by a nontrivial Ranking System [Tennenholtz 2004]

Ranked IIA

- Consider the statement: “An agent with votes from two weak agents should be ranked the same as one with a vote from one strong agent”.



Ranked IIA

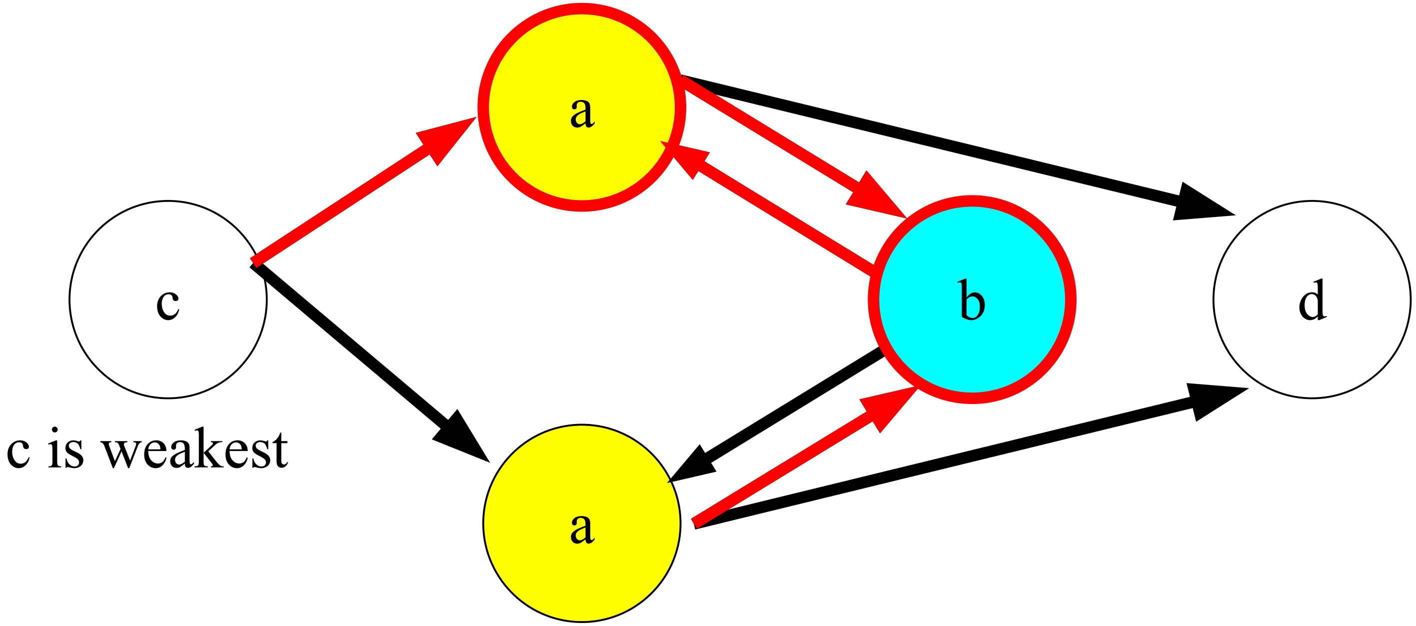
- We would like such comparisons to be *consistent*.
- That is, in every *profile* such as the one described in the previous slide we should decide >/</= consistently.
- This captures the Independence of Irrelevant Alternatives (IIA) for ranking systems.
- Can be seen as an ordinality requirement.
- Compare to Arrow's IIA axiom, which considers the name but not rank of the agents.

Impossibility

- **Theorem**: There exists no general Ranking System that satisfies Weak Transitivity and Ranked IIA.
- Proof: Constructive.
  - We assume existence of such ranking system and see graphs it cannot rank consistently.

Impossibility Proof – Part 1

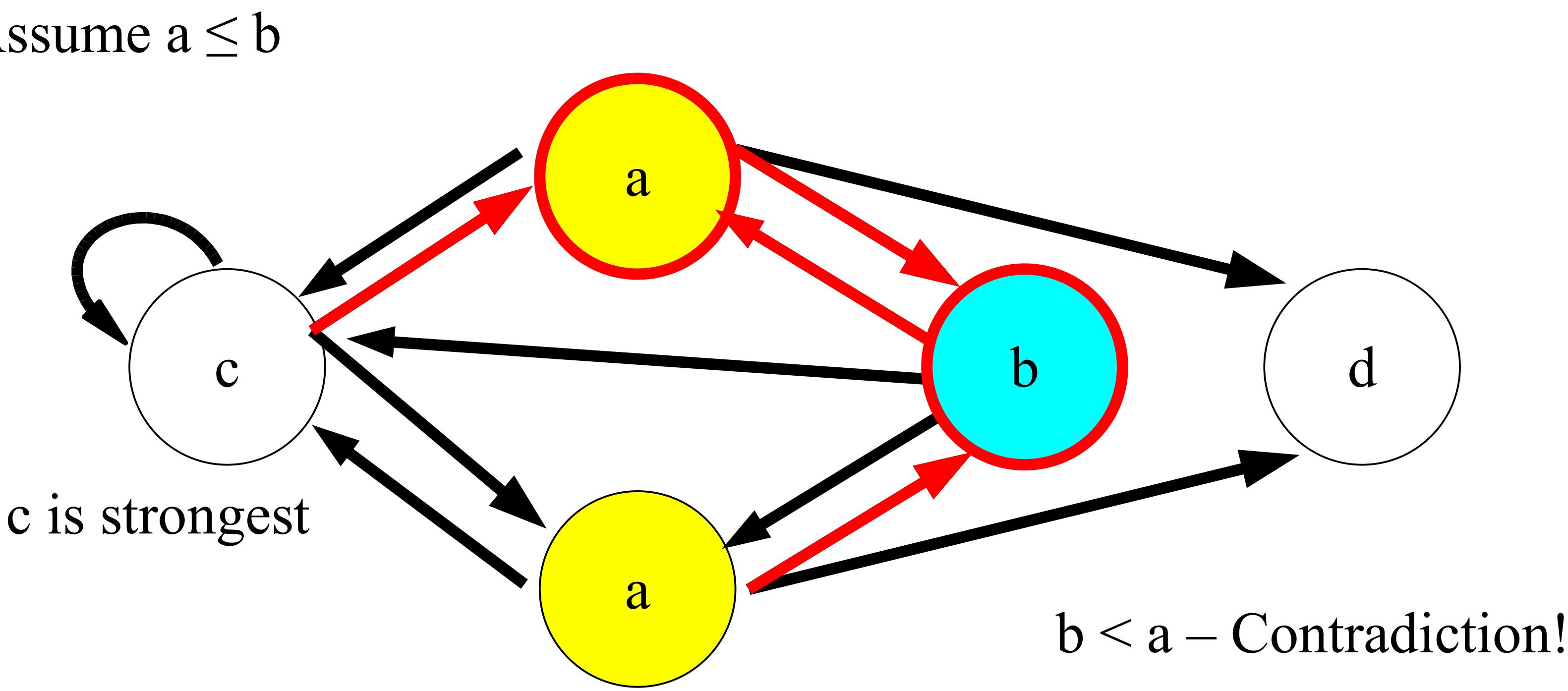
Assume b ≤ a



a < b – Contradiction!

→ A vertex with two equal predecessors is stronger than one with one weaker and one stronger predecessor.

Impossibility Proof – Part 2



→ A vertex with two equal predecessors is ***weaker*** than one with one weaker and one stronger predecessor.

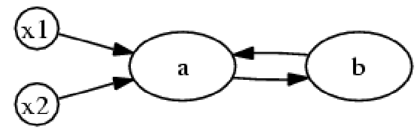
→ Contradiction to part 1. **QED**

Stronger Impossibility Results

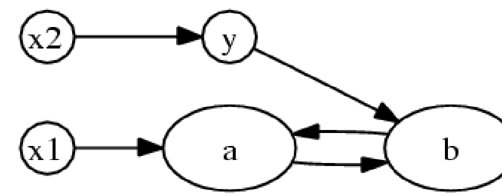
- Our impossibility result exists even in very limited domains:
  - Small graphs (4 agents are enough with Strong Transitivity).
  - Strongly connected graphs (as with PageRank).
  - Bipartite (buyer/seller) graphs.
  - Single vote per agent

One Vote Bipartite Proof

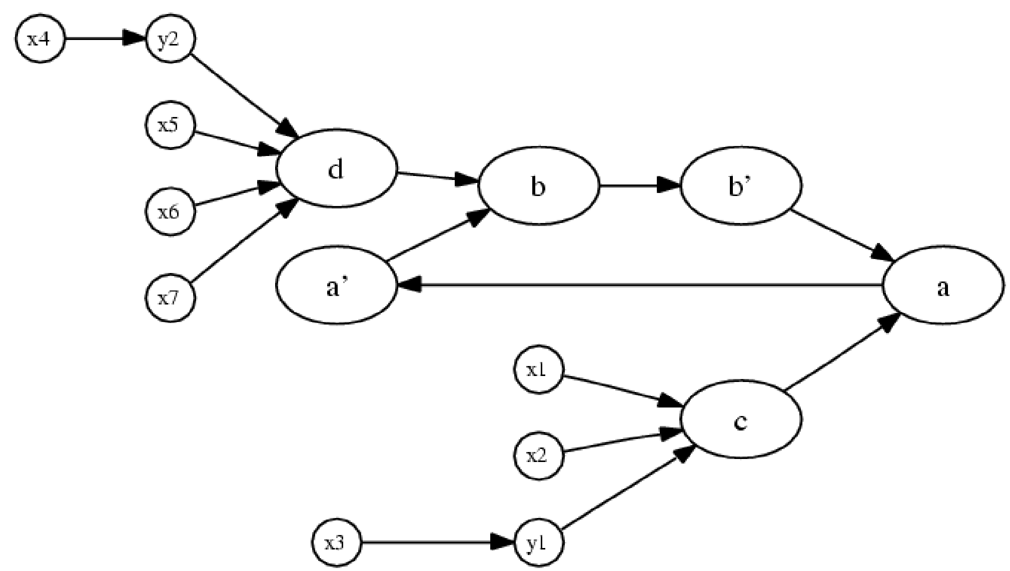
- In G_1: a(3) < b(1,1,2)
- In G_2: a(1,4) < b(2,3)
- In G_3: b(2,3) < a(1,4)
- Contradiction!



(a) Graph G_1



(b) Graph G_2



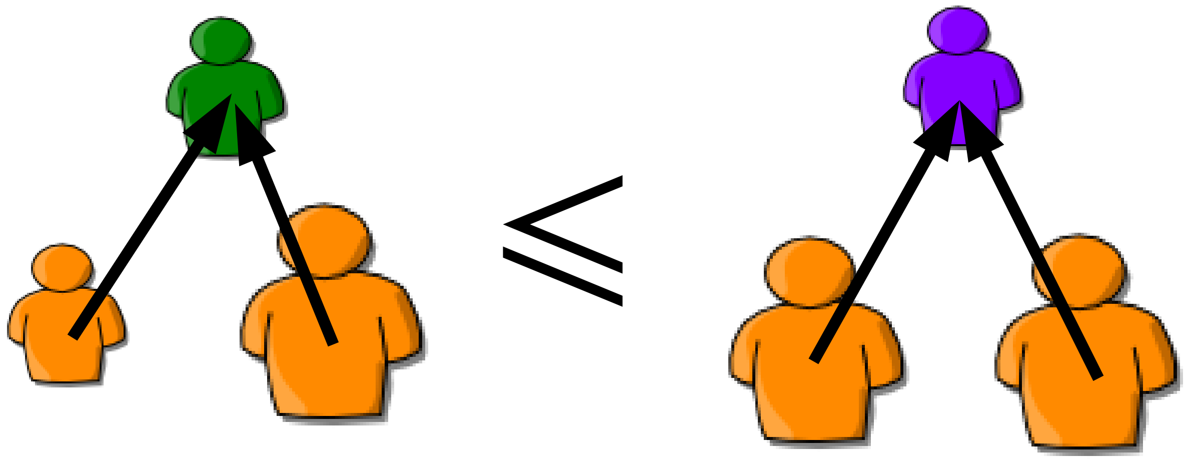
(c) Graph G_3

Transitive effects and IIA?

- We have proven that transitive effects and ranked IIA are incompatible.
- However, it turns out that under a different notion of transitivity these properties can be satisfied together.
- Moreover, the proposed ranking system is nontrivial and interesting.

Quasi-Transitivity

- We define the notion of *quasi-transitivity* as requiring only non-strict comparisons.
- A ranking system *F* satisfies *quasi- transitivity* if for every two vertices *x*,*y* where *F* ranks *x*'s predecessor set *P(x)* is weaker or equal to *P(y)*, then *F* must rank *x* weaker or equal to *y*.



Positive Result

- **Proposition:** There exists a nontrivial ranking system satisfying Ranked IIA and Quasi-Transitivity.
- The *recursive-indegree* ranking system can be defined using a simple and efficient algorithm:

$$v_1 \preceq_G^{RID_r} v_2 \Leftrightarrow \text{value}(v_1, r, \mathbf{0}) \geq \text{value}(v_2, r, \mathbf{0})$$

The *value* function

Procedure value(x, r, h) – returns numeric rank of node x under weight function r given previously seen nodes h:

1. Let $d := \begin{cases} 0 & |P(x)| = 0 \\ r(|P(x)|) & \text{Otherwise.} \end{cases}$

2. Let $h'(y) := \begin{cases} 0 & h(y) = 0 \wedge y \neq x \\ (n+1) \cdot h(y) + d & \text{Otherwise.} \end{cases}$

3. If $h(x) = 0$:

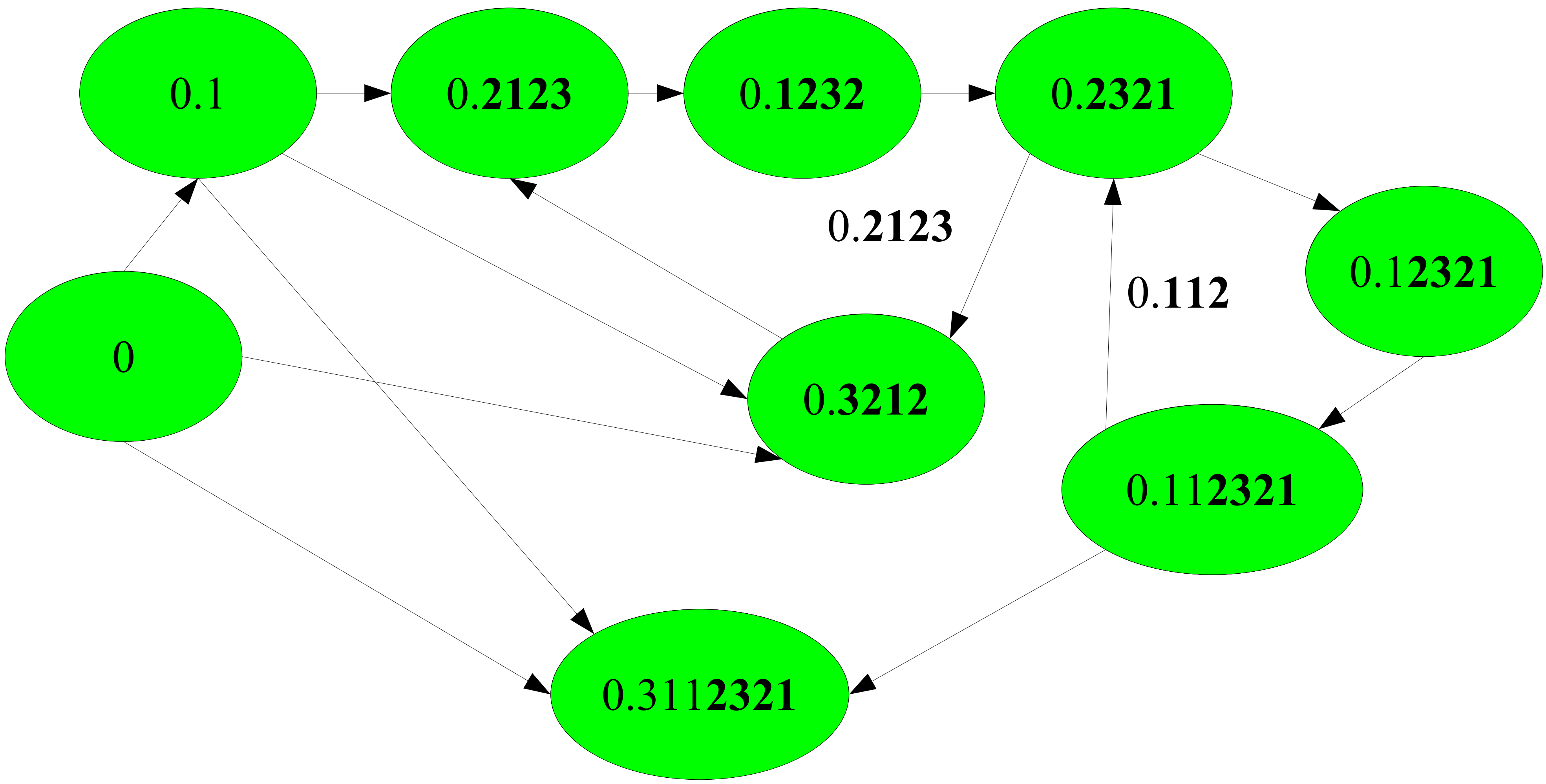
   (a) Return $\frac{1}{n+1} \left[d + \max(\{\text{value}(x, h', r) | p \in P(x)\} \cup \{0\})\right]$

4. Otherwise:

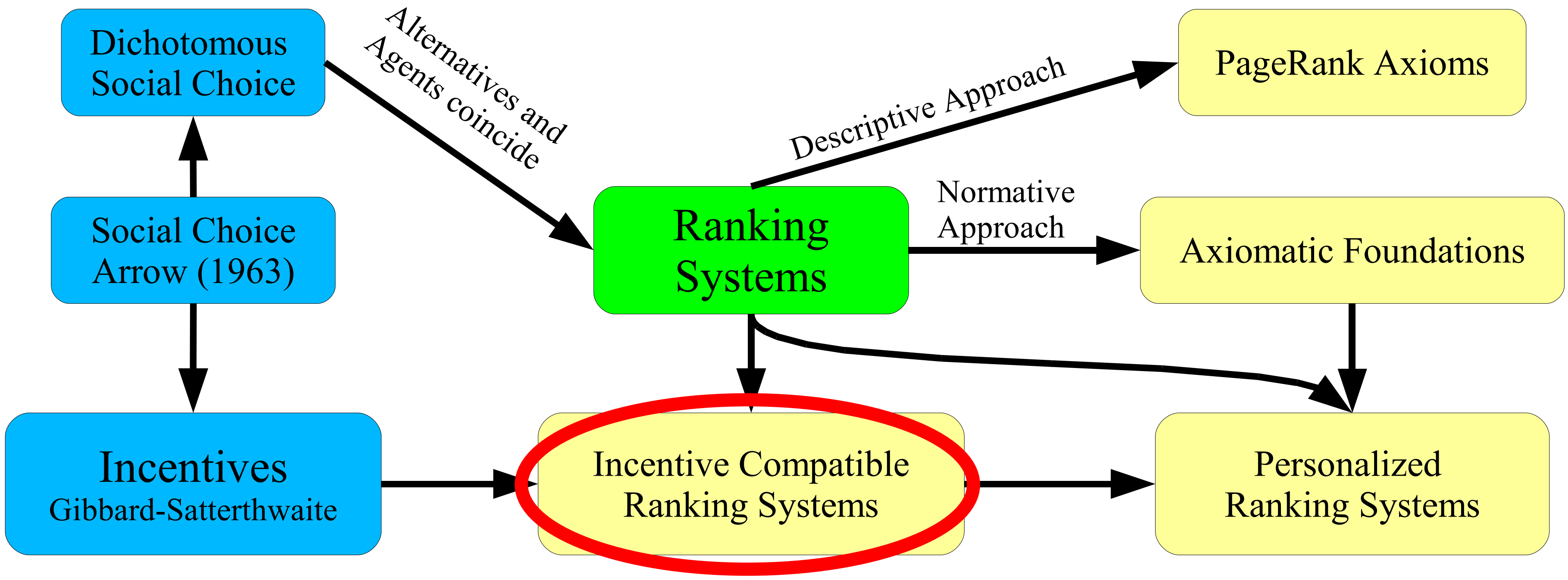
   (a) Let $m = \min\{(n+1)^k - 1 | (n+1)^k > h'(x)\}$.

   (b) Return $h'(x)/m$.

Example


0.1
0.2123
0.1232
0.2321
0.2123
0.12321
0.112
0
0.3212
0.112321
0.3112321

Research Map



Utility Function

- Formally, the utility function *u* for the agents maps for each agent count the number of agents ranked lower than the agent to a utility for that ranking:

$$u_n : \mathbb{N} \rightarrow \mathbb{R}$$

- The expected utility of an agent with *k* agents ranked strictly below it and *m* agents ranked the same is:

$$E[u_n] = u_n^*(k, m) = \frac{1}{m} \sum_{i=k}^{k+m-1} u_n(i)$$

Utility of a ranking

- Let $\preccurlyeq$ be the ordering of the agents of some ranking system F on some graph G=(V,E).
- The utility of agent *v* in graph *G* under ranking system *F* is:

$$u_G^F(v) = u_n^*(|\{u : u < v\}|, |\{u : u \simeq v\}|)$$

Affine utility

- A simple utility function is the identity function.
- Under the identity function we get:

$$u_n^*(k,m) = k + \frac{m-1}{2}$$

- Any *affine* utility function produces the same ordering over u(k,m), and thus is equivalent.
- We will focus on the special case of incentive compatibility under affine utility.

Incentive Compatibility

- Let *G=(V,E)* and *G'=(V,E')* be graphs that differ only in the outgoing edges from vertex *v*.
- A ranking system is *strongly incentive compatible*, if for every utility function *u*:

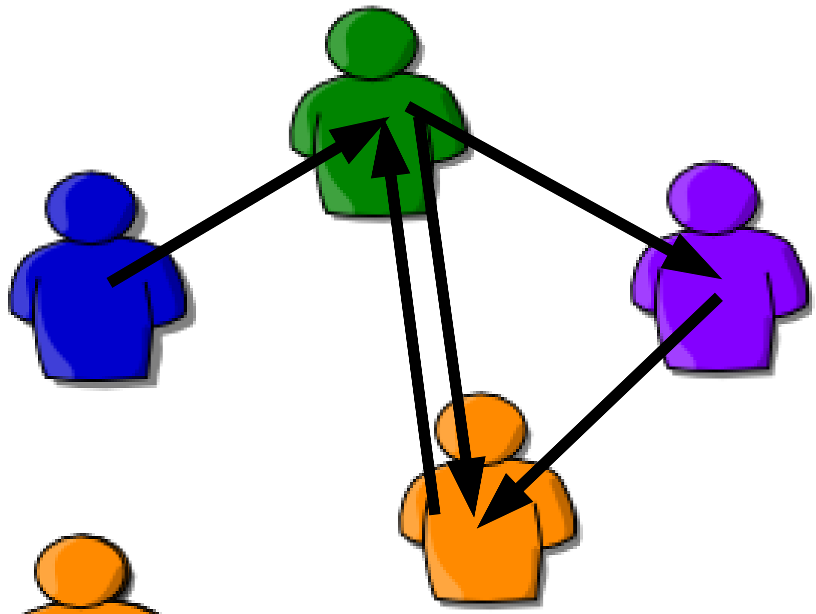
$$u_G^F(v) = u_{G'}^F(v)$$

- A ranking system is *incentive compatible* if for every *affine* utility function u':

$$u'^F_G(v) = u'^F_{G'}(v)$$

Example

- Approval voting: Count number of incoming edges.



Not Incentive Compatible!

Outline

- Related Work.
- Ranking Systems Setting.
- Incentive compatibility.
- Some Properties of Ranking Systems.
- Our Results.

Triviality

- A ranking system is *trivial* if the relative rank of two vertices is dependent only on their names and not on the graphs being ranked.
- A ranking system is *infinitely nontrivial* if there is an infinite series of disjoint graph pairs in which at least one common pair of vertices are ranked differently.

Non Imposition

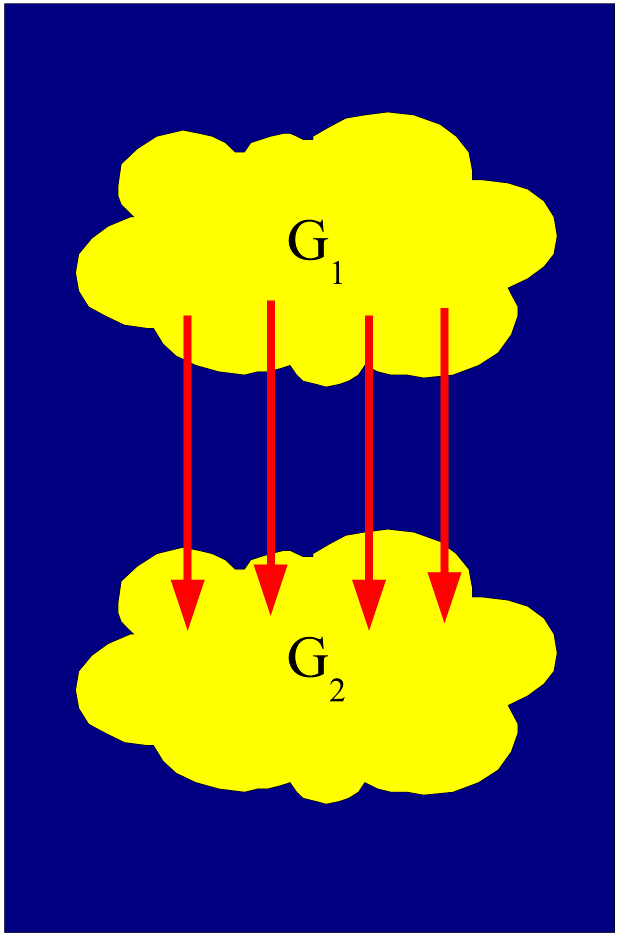
- Intuitively, we would like to see a stronger notion of nontriviality: We would like to see every strict ordering as a possible outcome.
- A Ranking System F is called Non-Imposing if for every vertex set V and for every strict ordering <, there exists a graph G=(V,E) such that F ranks G according to <.

Minimal Fairness

- A weaker notion than Isomorphism is Minimal Fairness, which requires global ties in trivial graphs.
- A ranking system is *weakly minimally fair* if it ranks all agents equally when there are no edges in the graph.
- A ranking system is *minimally fair* if it furthermore ranks all agents equally when the graph is a complete clique.

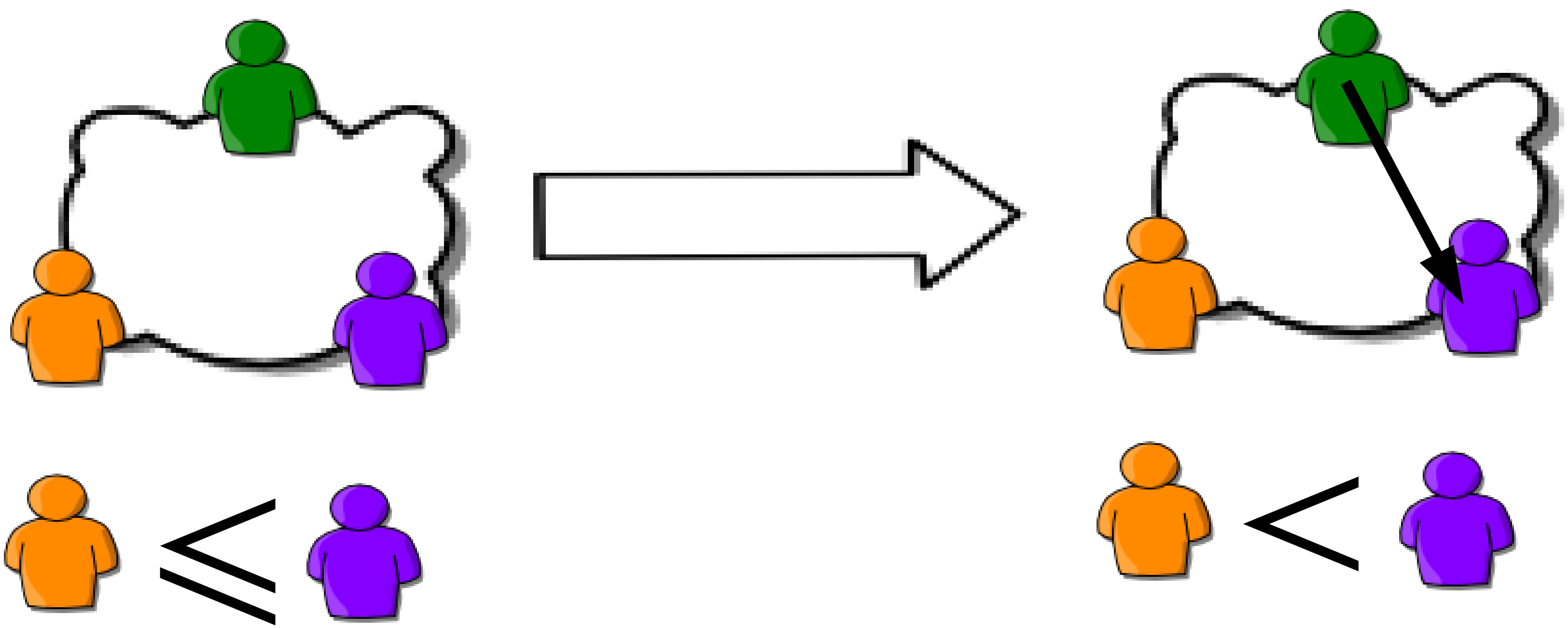
Union Condition

- The *weak union condition* requires that in a graph with two completely separate segments, each segment should be ranked the same as if it were on its own.
- The *strong union condition* further requires that if there are edges from only one segment to the other, the source segment should be ranked the same as if it were on its own.



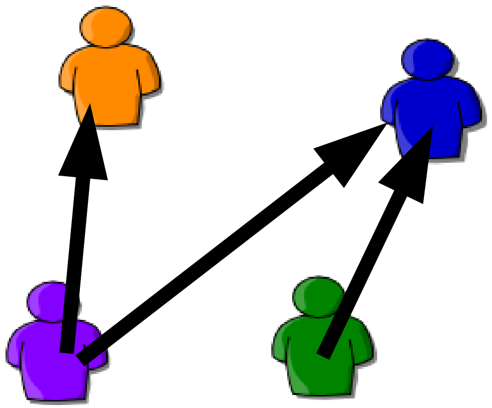
Positive Response

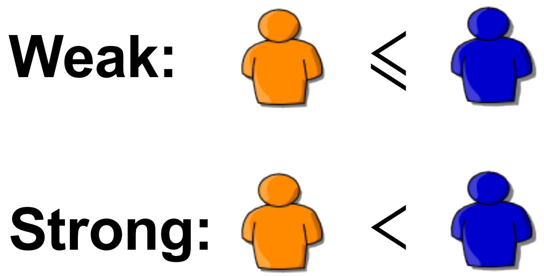
- A ranking system satisfies (*strong) positive response*, if adding an edge will not decrease(increase) the relative ranking of the edge's target.



Monotonicity

- A ranking system satisfies weak monotonicity if an agent voted by a superset of the voters of another will be ranked at least as strong.
- A ranking system further satisfies strong monotonicity if an agent voted by a strict superset of the voters of another will be ranked strictly stronger.





Outline

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Possibility without Minimal Fairness

- If we do not assume minimal fairness, we can find a ranking system that satisfies:
  - Strong Incentive Compatibility
  - Strong Positive Response
  - Strong Union Condition
  - Infinite Nontriviality
- The system ranks all agents based on a fixed order, swapping v_2 and v_3 if the edge (v_1,v_3) exists in the graph.

Classification under Weak Minimal Fairness

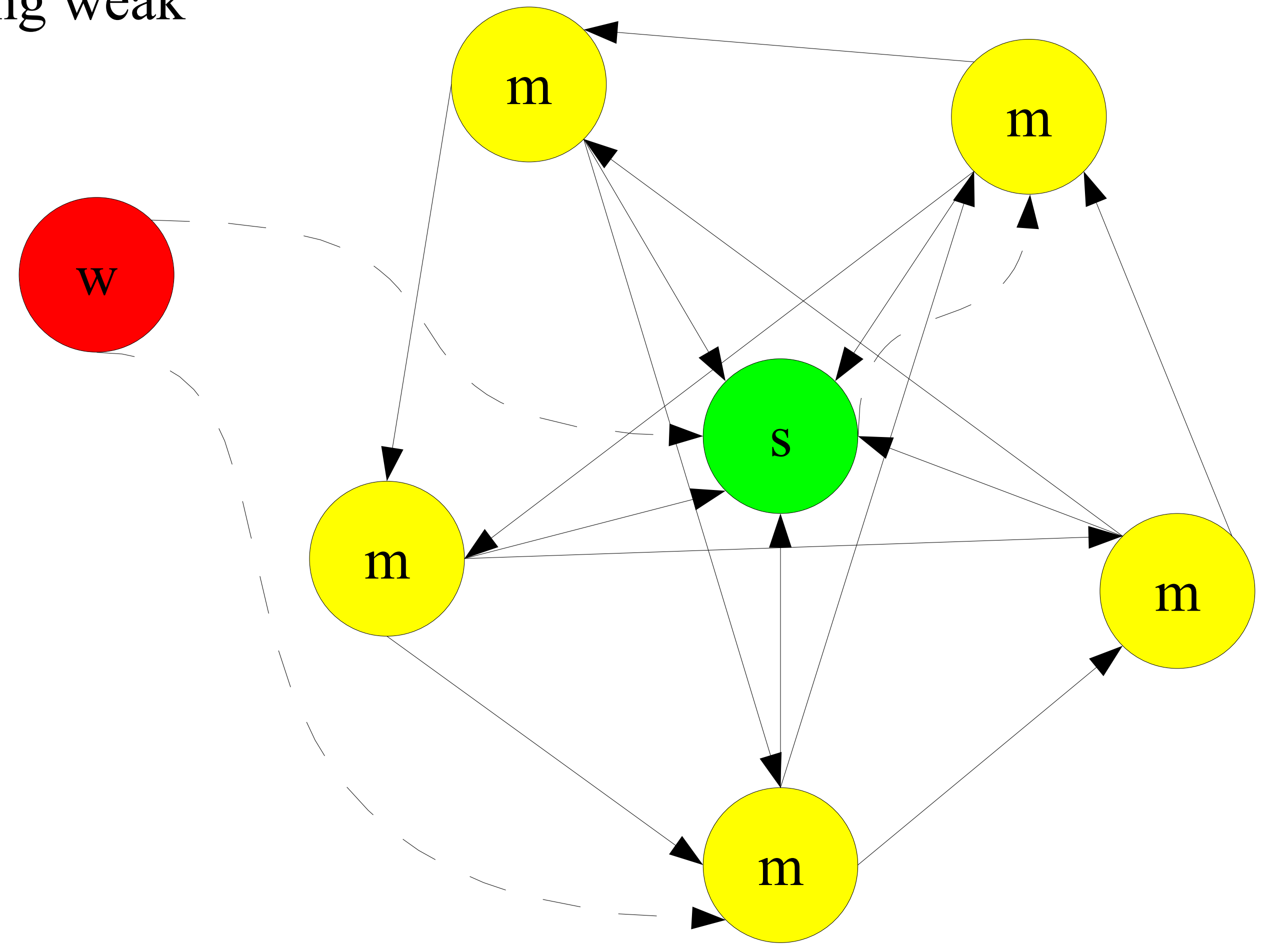
- There exist incentive compatible, weakly minimally fair, infinitely nontrivial ranking systems that satisfy each of the following conditions, but no incentive compatible, weakly minimally fair, nontrivial ranking systems that satisfy any two of these:
  - The weak union condition;
  - Weak positive response; and
  - Weak monotonicity

Classification under Weak Minimal Fairness (cont.)

- There exists no incentive compatible, nontrivial, weakly minimally fair ranking system that satisfies either one of:
  - Strong Incentive Compatibility
  - The strong union condition
  - Strong positive response; and
  - Strong monotonicity

Partial Proof Sketch

An incentive-compatible ranking system satisfying weak monotonicity



Non Imposition and Incentive Compatibility

- There is no general incentive compatible non-imposing ranking system.
- However, if we consider the setting of graphs with exactly three vertices, there exist such ranking systems.
- These systems further satisfy weak positive response and minimal fairness.

A Non-Imposing Ranking System for Three Agents

- All such systems can be defined by the table below.
- The difference between the systems are how votes for both others and none of the others are interpreted

		$v_0 \rightarrow v_1$	$v_0 \rightarrow v_2$
$v_2 \rightarrow v_0$	$v_1 \rightarrow v_2$	$\simeq$	$v_1 \prec v_0 \prec v_2$
	$v_1 \rightarrow v_0$	$v_2 \prec v_1 \prec v_0$	$v_1 \prec v_2 \prec v_0$
$v_2 \rightarrow v_1$	$v_1 \rightarrow v_2$	$v_0 \prec v_2 \prec v_1$	$v_0 \prec v_1 \prec v_2$
	$v_1 \rightarrow v_0$	$v_2 \prec v_0 \prec v_1$	$\simeq$

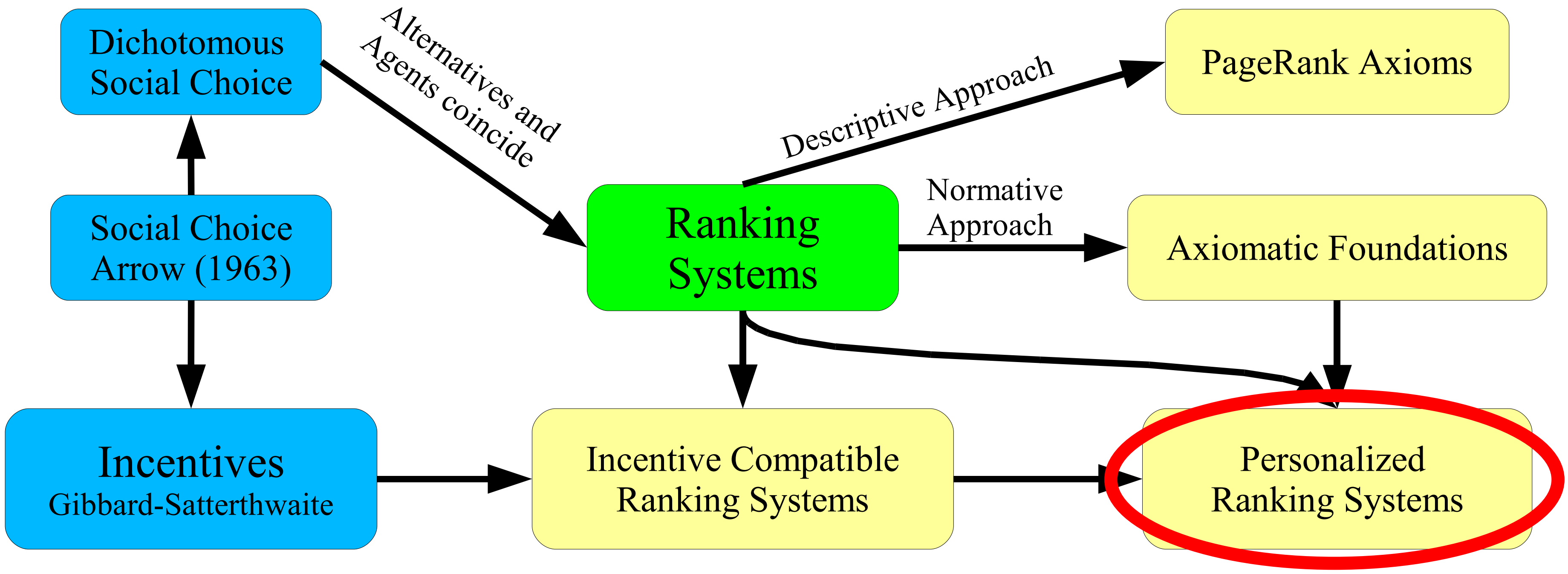
Relaxing Incentive Compatibility

- In an extension of this work, we have quantified levels of incentive compatiblity of ranking systems:
  - Under strong monotonicity, there will always be a graph with a deviation that will improve an agent's rank by at least n/2.
  - There are non-imposing ranking systems that are nearly incentive compatible.

Summary

- Incentive compatibility in Ranking Systems was defined.
- Full classification of incentive compatible ranking systems was provided in the terms of basic properties of ranking systems.
- An additional postive result was given for a special case non-imposing ranking systems.
- Relaxing incentive compatibility leads to more positive and negative results.

Research Map



Personalized Ranking Systems

- The “client” of the ranking system may also be a participant.
- Examples:
  - Social Networks
  - C2C commerce sites (eBay)
  - Trust (PGP).
- It is useful to generate a personalized ranking for each individual.
- Many impossibility results are reversed.

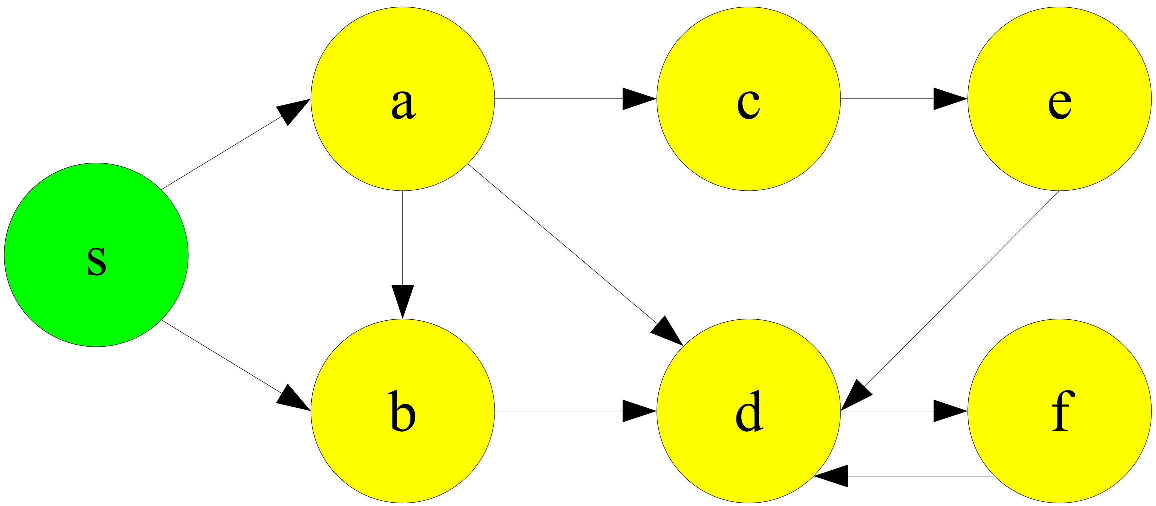
What is a personalized ranking system?

- A personalized ranking system is like a general ranking system, except:
  - Additional parameter: the *source*, i.e. the agent under whose perspective we're ranking.
  - Defined only on the graphs where the source *s* is a *root*, that is there is a directed path from *s* to all vertices.
    - Usually we simply assume the graph is strongly connected.

Examples of PRSs

- **Distance rule** - rank agents based on length of shortest path from s.
- **Personalized PageRank** with damping factor *d* - The PageRank procedure with probability *d* of restarting at vertex *s*.
- **α-Rank** - Rank based on distance, but break ties based on lexicographic order on predecessor rank.

Example of Ranking



Distance

s, a=b, c=d, e=f

α-Rank

s, b, a, d, c, f, e

Personalized PageRank

(d=0.2)

d, **s**, f, b, a, c, e

(d=0.5)

s, b, a, d, f, c, e

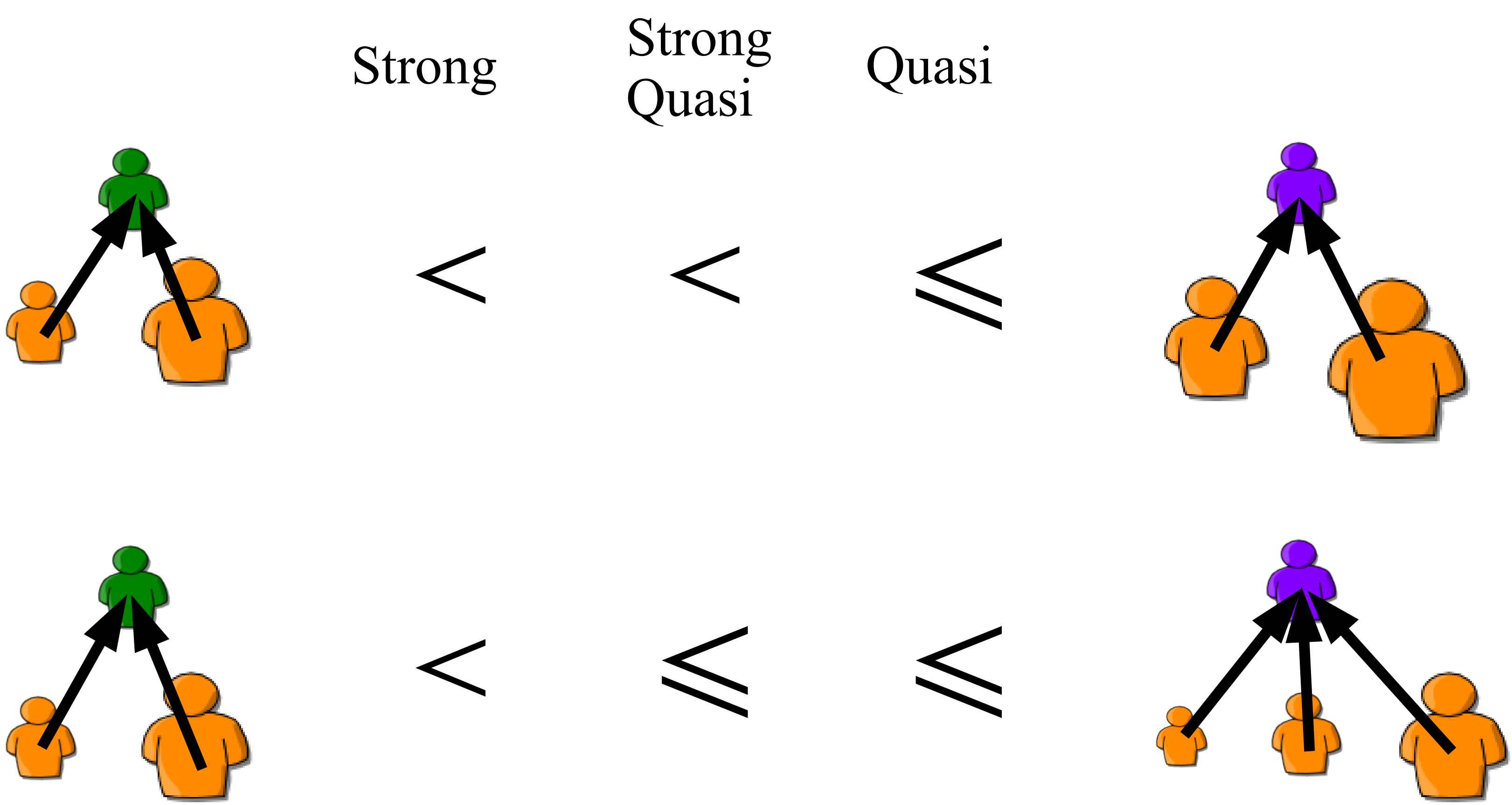
Properties of PRSs

- A PRS satisfies *self-confidence* if the source *s* is ranked stronger than all other vertices.

- The following properties from general ranking systems could be adapted to PRSs.
  - Strong/Quasi/Weak transitivity
  - Ranked IIA
  - Strong Incentive Compatibility

- In every case, we require the property to be satisfied by all vertices except *s*.

Types of Transitivity



New type of Transitivity

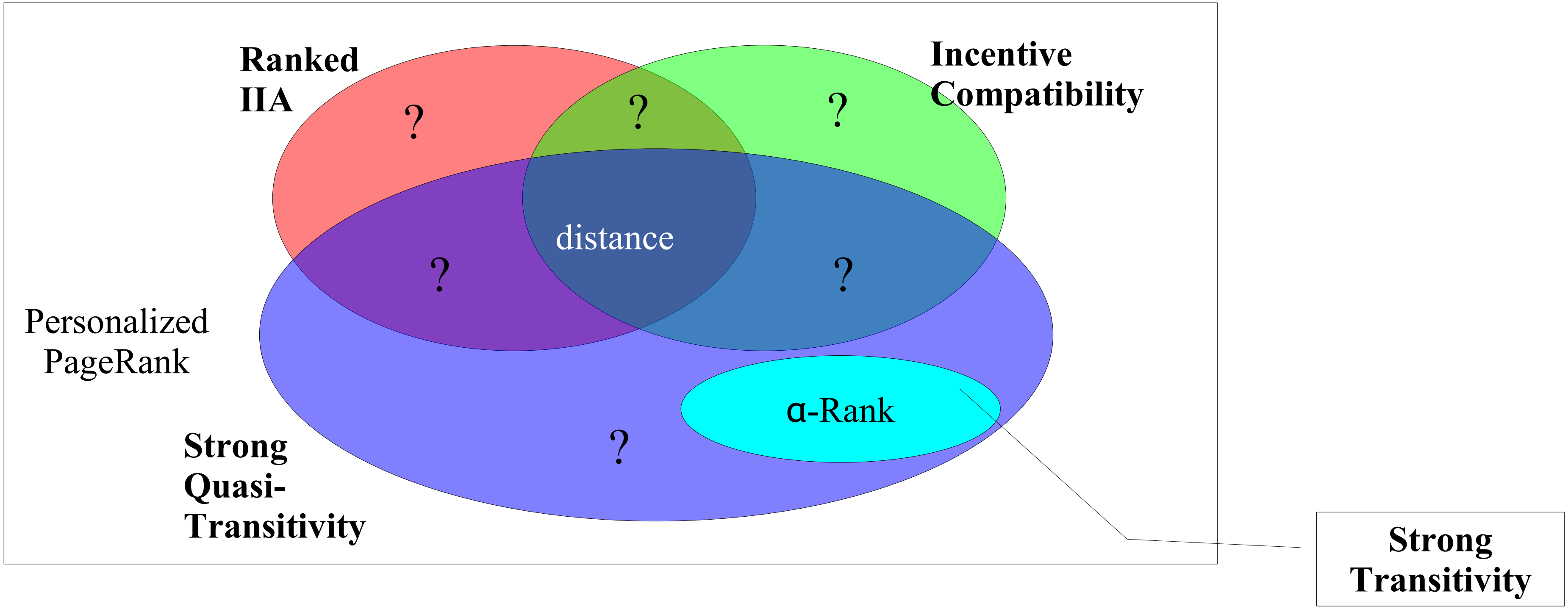
- Assume a ranking system F and two vertices x,y (excluding the source) with a mapping f from P(x) to P(y) that maps each vertex in P(x) to one at least as strong in P(y).
  - **Quasi-transitivity**: $y \precsim x$.
  - **Strong Quasi transitivity**: Furthermore, if *all* of the comparisons are strict: $y \prec x$.
  - **Strong transitivity**: Furthermore, if *at least one* of the comparisons is strict or f is not onto: $y \prec x$.

Classification of PRSs

- **Proposition:** The **distance PRS** satisfies self confidence, ranked IIA, strong quasi transitivity, and strong incentive compatibility, but does not satisfy strong transitivity.
- **Proposition:** The **Personalized PageRank** ranking systems satisfy self confidence iff d>1/2. Moreover, Personalized PageRank does not satisfy quasi transitivity, ranked IIA or incentive compatibility for any damping factor.
- **Proposition:** The **α-Rank PRS** satisfies self confidence and strong transitivity, but does not satisfy ranked IIA or incentive compatibility.

Summary

PRS	Distance	P. PageRank	α-Rank
Self Confidence	YES	for d> 1/2	YES
Ranked IIA	YES	NO	NO
Transitivity	strong quasi	none	strong
Incentive Comp.	strong	none	none

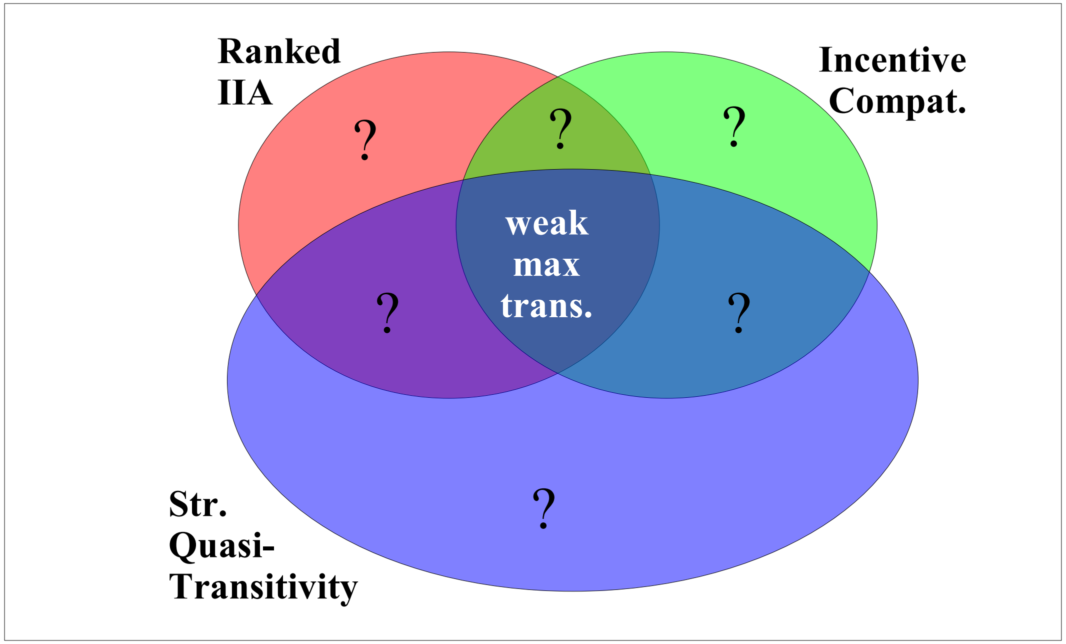


Maximum Transitivity

- Assume a ranking system F and two vertices x,y (excluding the source).
- If F satisfies *weak maximum transitivity*, then if the strongest predecessor of x is at least as strong as the the strongest predecessor of y, then y $\precsim$ x.
- **Fact:** If F satisfies weak maximum transitivity, then F must be a refinement of the distance ranking system.

Classification Theorem

- **Theorem:** Let *F* be a PRS that satisfies self confidence, strong quasi transitivity, RIIA and strong incentive compatibility.
  Then, *F* satisfies weak maximum transitivity.
- **Corollary:**
  F is a refinement of the distance ranking system.



Relaxing the Axioms

- All axioms are required for the previous result.
- If we relax any axiom, the system no longer satisfies weak maximum transitivity.
- In particular there are artificial systems with the following properties:

Self Confidence	YES	NO
Ranked IIA	YES	YES
Str.Quasi-Trans	NO	YES
Inc. Comp	YES	YES
Weak max-Trans	NO	NO

Relaxing Ranked IIA

- The Path Count PRS ranks vertices based on the number of directed paths each vertex has from the source.
- **Proposition:** The path count PRS has the following properties:

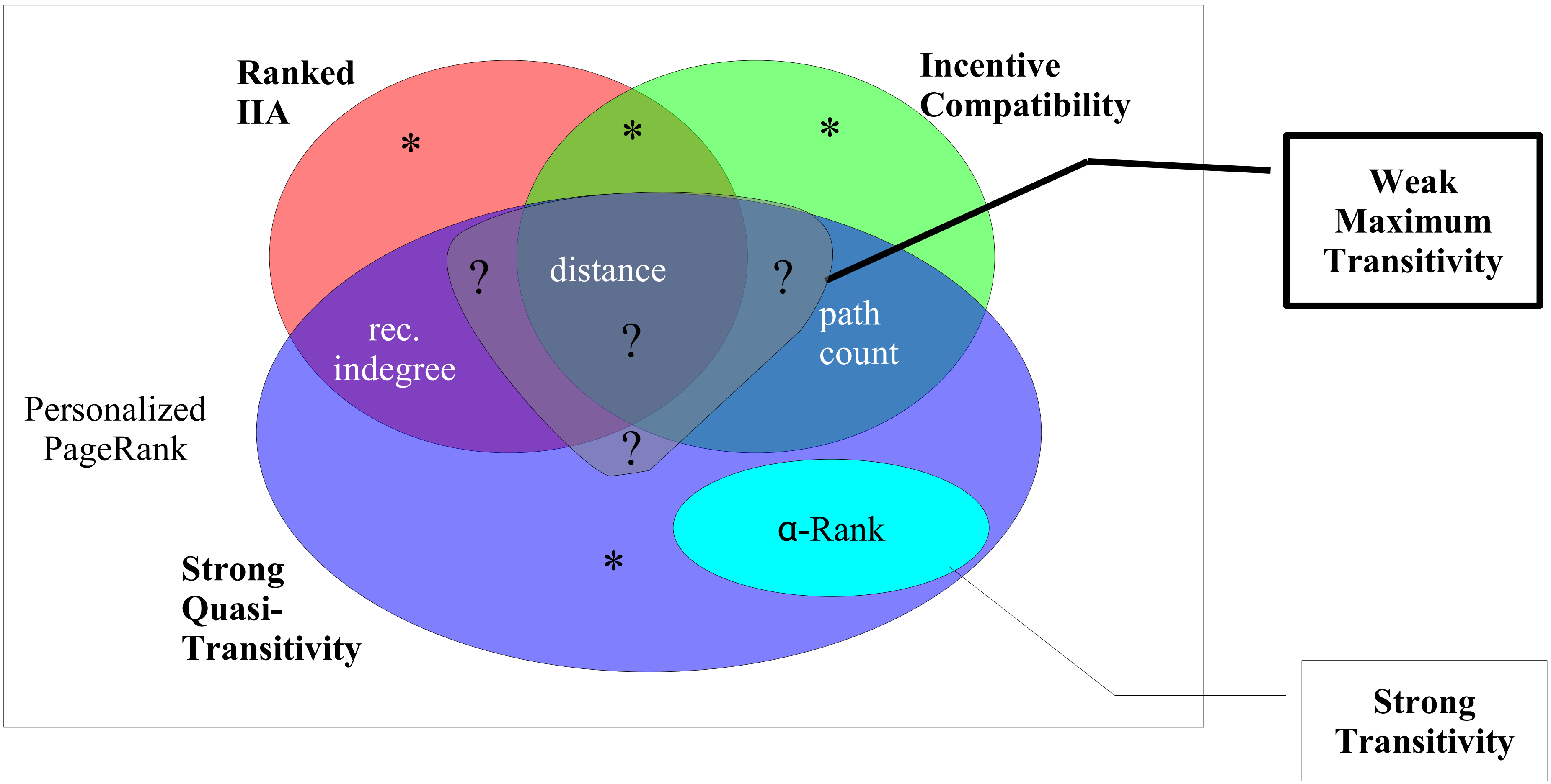
Self Confidence	YES
Ranked IIA	NO
Str.Quasi-Trans	YES
Inc. Comp	YES
Weak max-Trans	NO

Relaxing Incentive Compatibility

- The recursive in-degree ranking system can be adapted to the personalized setting by giving the source vertex a maximal value, as if it has in-degree n+1.
- **Proposition:** The recursive in-degree PRS has the following properties:

Self Confidence	YES
Ranked IIA	YES
Str.Quasi-Trans	YES
Inc. Comp	NO
Weak max-Trans	NO

Personalized Ranking Systems -- Summary



* Artificial Ranking Systems

Summary

- In the Normative Approach, we have seen both impossibility and possibility results.
- We have seen the impossibility of incentive compatible ranking systems.
- We have applied this approach to personalized ranking systems, with very positive results.